Journal of Statistical Physics, Vol. 26, No. 1, 1981

The Influence of Noise on the Logistic Model

G. Mayer-Kress[1] and H. Haken[1]

Received August 6, 1980; revised January 29, 1981

The behavior of the logistic system which is generated by the function $f(x) = ax(1-x)$ changes in an interesting way if it is perturbed by external noise. It turns out that the chaotic behavior which was predicted by Li and Yorke for orbits of period 3, becomes visible and that a sequence of mergence transitions occurs at the critical parameter. The change of the invariant probability density and the Lyapunov exponents are examined numerically. The power spectrum for the period 3 orbit for different fluctuations is calculated and a recursion formula for the time evolution of the probability density is presented as a discrete-time analog of a Chapman–Kolmogorov equation.

KEY WORDS: Dynamical system; chaos; bifurcations; attractor; Lyapunov exponents; fluctuations; Chapman–Kolmogorov equation.

1. INTRODUCTION

The qualitative features of macroscopic systems which undergo transitions to turbulent behavior have been simulated by many model systems. So the Lorenz equations[1] and their more sophisticated modifications by Curry[2] have served as a model for hydrodynamic turbulence. Their connection with irregular laser pulses has been shown by Haken and Wunderlin.[3,4]

Other models have been constructed for chaotic oscillations in (bio)-chemical reactions and in populations of ecological systems, e.g., by Rössler[5,6] and May.[7]

While these models usually consist of a set of nonlinear differential equations, there is a different approach which considers iterates f^n of a certain map f. This function f can either be visualized as a Poincaré map[8] or as a stroboscopic or flash light portrait of the trajectory of a continuous time system.[9]

[1] Institut für theoretische Physik, Universität Stuttgart, Pfaffenwaldring 57/4, D-7000 Stuttgart 80, Federal Republic of Germany.

0022-4715/81/0900-0149$03.00/0

These discrete-time dynamical systems have among others the advantage of greater simplicity with respect to numerical calculations as well as to mathematical analysis. So for instance it has been rigorously proved by different authors[10-13] that already maps of the "quadratic-type" of the one-dimensional interval onto itself show chaotic properties. It has been also proved for a class of functions that there exists an invariant measure which is absolutely continuous with respect to the Lebesque measure.

In numerical experiments one can "see" this when the histogramatic measure is different from zero in the whole invariant interval. Furthermore the (measure-theoretic) entropy of those systems becomes positive, which can be calculated via the Lyapunov characteristic exponents. This also means that we have a sensitive dependence on initial conditions.

An example where "chaos" which was predicted by a mathematical theorem could not be seen in computer experiments is the well-known theorem of Li and Yorke which states for maps of an interval onto itself: "period 3 implies chaos."[14] There it was proved that in the presence of an orbit of period 3 there exist orbits of arbitrarily high periods and a nondenumerable set of points, which have aperiodic orbits. [Here period-3 means $f^3(x_0) \leqslant x_0 < f(x_0) < f^2(x_0)$.]

This result was generalized to orbits of period $\neq 2^n$ by Oono.[15] What one really sees on the computer is often not chaos but periodic orbits which are just as regular and globally stable as any other period. The reason that one cannot find these aperiodic orbits numerically is that they are repelling and also of Lebesgue-measure zero. It seems to be a common feature of these systems that if one increases the relevant parameter, the system undergoes a sequence of period-doubling bifurcations until above a critical parameter value a chaotic regime begins. This chaotic regime is, however, not homogeneous but interrupted by "parameter windows" in which stable periodic solutions are present. These "windows" have been seen not only in one-dimensional models (cf. Fig. 13 and Ref. 16), but also, e.g., in the two-dimensional discrete Hénon attractor[17] and even in the three-dimensional Lorenz attractor.[18-20] Since one cannot predict exactly for which parameters such stable periodic orbits exist, one cannot distinguish by numerical calculations between a periodic orbit of very long period and a truly aperiodic orbit.

On the other hand it has been proved that aperiodic solutions are not too rare, in the sense that there exists for a certain class of functions a set of parameter values with positive Lebesgue measure, for which no stable periodic orbits exist. With the help of renormalization-group techniques one can even predict where these parameter values can be expected.[21,22]

From a physical point of view it seems natural to ask how much of the qualitative behavior of these model systems survives in nature, where small perturbations are permanently present.[23-26] So one can introduce fluctuations, not in order to produce chaos but to check how stable a deterministic dynamical system is against small perturbations. What we find is that for the logistic system the stable periodic orbits with periods $\neq 2^n$ disappear much faster than orbits of stable period 2^n and in a way which is qualitatively different. The threshold for the size of the fluctuations for which, e.g., the Lyapunov exponents become positive is an order of magnitude smaller in the first case. The transition to chaos manifests itself in a characteristic change of the probability density, the Lyapunov exponents, and the autocorrelation function (power spectrum).

2. DEFINITION OF THE MODEL SYSTEM CONSIDERED

We examine the influence of random perturbations on the logistic system, which is defined by the family of functions

$$f_a : [0,1] \to [0,1] \qquad x \mapsto ax(1-x) \tag{1}$$

where $a \in [0,4]$. (In the following we put: $f := f_a$.) An orbit (or trajectory) for this system is given by a sequence of numbers $x_n \in [0,1]$, $n \in \mathbb{N}_0 := \{0, 1, 2, \ldots\}$, such that

$$x_{n+1} = f(x_n) \tag{2}$$

In order to simulate the fluctuations, we add to each x_n, $n > 0$ a pseudorandom number ξ_n. The random numbers were equally distributed in the interval $I_\beta := [-\beta, \beta]$. They had zero mean and standard deviation $\sigma = \beta/\sqrt{3}$.

So the perturbed system is given by sequences $(\tilde{x}_n)_{n \in \mathbb{N}_0}$, where

$$\tilde{x}_0 = x_0, \qquad \text{and} \qquad \tilde{x}_n = f(\tilde{x}_{n-1}) + \xi_n \quad \text{for } n > 0 \tag{3}$$

Since the numbers ξ_n are the outcome of a pseudo-random-number generator of a computer, they can also be interpreted as a dynamical system with some deterministic generating function

$$g : I_\beta \to I_\beta \qquad \xi_n \mapsto \xi_{n+1} \tag{4}$$

In this way we are in fact considering a two-dimensional dynamical system

$$F : I_B(\beta) \times I_\beta \to I_B(\beta) \times I_\beta \quad (\tilde{x}_n, \xi_n) \mapsto (\tilde{x}_{n+1}, \xi_{n+1}) \tag{5}$$

where $I_B(\beta) \subset [0,1]$ is the basin of the system defined below. This can also be interpreted as the interaction of the two systems f and g.

3. CONDITION FOR THE BOUNDEDNESS OF THE LOGISTIC SYSTEM

The domain of the logistic function (1) is restricted to the unit interval because the iterations of starting values outside of [0, 1] diverge to minus infinity. That is also the reason why the fluctuations in (3) must have an upper bound, if the system should be confined to a finite interval. The fact that we are not allowed to admit fluctuations of arbitrary largeness is not in contradiction with intuition since we are dealing with a system which corresponds, e.g., in ecology to a finite population which of course will die out if there is a large enough fluctuation, i.e., a fatal catastrophe.

For the upper bound $\beta_{\max}$ of the fluctuations ξ_n we have the following:

Proposition. Let $\xi_n \in [-\beta, +\beta]$ for $\beta \geqslant 0$ and for all $n \in \mathbb{N}$ let $a \in [3,4]$. Then there exists

$$\beta_{\max} \geqslant \frac{1}{4a}\left[2a - a^2 - 4 + 2(3a^2 - 4a + 4)^{1/2}\right] \tag{6}$$

and a basin $I_B(\beta) \subset [0,1]$ such that for all $\tilde{x}_0 \in I_B(\beta)$ and for all $\beta \leqslant \beta_{\max}$ the sequence $(\tilde{x}_n)_{n \in \mathbb{N}}$ of Eq. (3) is contained in $I_B(\beta)$.

Proof. See Appendix A.

4. TIME DEVELOPMENT OF THE PROBABILITY DENSITY

From the point of view of statistical mechanics, only the "average" influence of the system g onto the system f is interesting. Furthermore a realistic system should be independent of the actual choice of a specific random sequence. Therefore one should only consider the ensemble, in which the systems (3) have the same generating function f but different sequences $(\xi_n)_{n \in \mathbb{N}}$. Now we can define the ensemble average $\langle r \rangle$ of some function r by

$$\langle r \rangle := \int_{-\infty}^{\infty} R(\xi) r(\xi) d\xi \tag{7}$$

where R denotes the density function of the random numbers ξ_n. Thus we get a recursion formula for the probability density $\tilde{P}(x, n+1)$ that the system, starting at x_0, is at the point x after $n+1$ iterations. It is given by (cf. Appendix B)

$$\tilde{P}(x, n+1) = \int_{-\infty}^{\infty} R(f(y) - x)\tilde{P}(y, n) dy \tag{8}$$

In our case, where the ξ_n are equally distributed, we have

$$R(x) = (1/2\beta)\chi_{I_\beta}(x) \tag{9}$$

where

$$\chi_I(x) := \begin{cases} 1 & \text{if } x \in I \\ 0 & \text{if } x \notin I \end{cases}$$

is the characteristic function of the interval I. For the density (9), the recursion formula (8) becomes

$$\tilde{P}(x, n+1) = \frac{1}{4a\beta} \int_{-\beta}^{\beta} \frac{dz}{u(z+x)} \times \left[\tilde{P}\left(\tfrac{1}{2} - u(z+x), n\right) + \tilde{P}\left(\tfrac{1}{2} + u(z+x), n\right) \right] \quad (10)$$

where we have put $u(x) := (1/4 - x/a)^{1/2}$.

In our numerical calculations it appeared that the probability-density of the system (3) became stationary after some 10^6 iterations and seemed to be independent of the initial values of x_0 and ξ_1.

5. THE PROBABILITY DENSITY OR "HISTOGRAMATIC MEASURE"

The histogramatic measure gives a frequency distribution of a trajectory through a certain starting point in the unit interval. It is invariant under the transformation (3) if: $\tilde{P}(.,n+1) = \tilde{P}(.,n) := \tilde{P}^*$ in Eq. (8). For a system with stable orbit $x_0, \ldots, x_{n-1}$ of period n, it is given by

$$\tilde{P}^*(x) = \sum_{i=0}^{n-1} \delta(x - x_i) \quad (11)$$

On the computer this is approximated after $m > n$ steps and with a resolution of $1/k$ of the unit interval (i.e., the unit interval is equally partitioned into k subintervals I_i) by the histogram

$$\tilde{P}^*(x, m, k) = \frac{m}{nk} \sum_{i=1}^{k} \sum_{j=0}^{n-1} \chi_{I_i}(x) \chi_{I_i}(x_j) \quad (12)$$

This is a function with n "sharp" peaks. Consequently, an aperiodic orbit or chaos is represented by a histogramatic density which is different from zero on "large subsets" of the unit interval. Lorenz called a system "semiperiodic" if it possesses an attractor which is aperiodic but looks periodic.[27] To define a semiperiodic orbit let $(x_k)_n := (x_k, x_{k+n}, x_{k+2n}, \ldots)$ and denote by "range of $(x_k)_n$" the closure of the set of elements of the sequence $(x_k)_n$. Then an aperiodic sequence $(x_n)_{n \in \mathbb{N}}$ is called *semiperiodic* of period n if the ranges of $(x_k)_m$ where $0 \leqslant k < n$ are disjoint for $m = n$ but overlap for $m > n$. For a typical semiperiodic system the histogram consists of n different "broad" peaks and is zero elsewhere, i.e., the system jumps periodically to n different "islands" in the unit interval. In the continuous time case this kind of behavior (which is also characterized by a power spectrum which consists of broadband noise and sharp peaks) is known as "phase coherence".[28]

Lorenz described semiperiodic behavior of a discrete deterministic system (2), where transitions to successively lower semiperiodicities—which he called "reverse bifurcations"—occur, as the relevant parameter a is increased beyond the critical value. The a_n at which these reverse bifurcations take place converge to a certain limit a_c in a way which is quite similar to the sequence of subharmonic bifurcations[29,30] where the ratio $(a_{n+1} - a_c)/(a_n - a_c)$ converges to a fixed limit for $n \to \infty$. During such a reverse bifurcation the disjoint components of the support of the invariant measure merge pairwise similar to Figs. 1 and 2. In spite of this apparent similarity, the "mergence transitions" which we observe for the perturbed system (3) represent a different behavior. Here we fix the parameter a at a value for which the unperturbed system (1) has a maximally stable periodic orbit with period p. If $p = m \cdot 2^n$, where $m, n \in \mathbb{N}$, we observe a sequence of mergence transitions when we increase the fluctuation size β as the relevant parameter. During this procedure the orbits are always semiperiodic of successively lower periods, while between two reverse bifurcations—in the sense of Lorenz—many different periodic or aperiodic attractors might exist. In Table I we have listed the first n values β_i where mergence transitions take place when we choose different parameters. They correspond to values at which system (1) has stable orbits of period $2^6, 2^\infty, 3 \cdot 2^\infty$, respectively. For the case of (semi) periodic orbits of period $p = 2^m$ (to

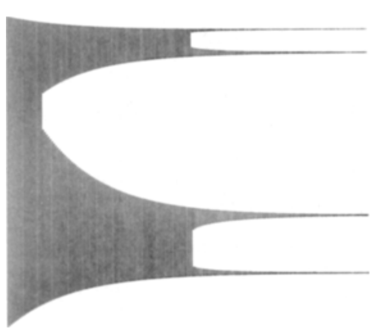

Fig. 1. Support of the invariant measure (drawn as vertical lines) versus the logarithm of β (which is equal to the standard deviation of the fluctuations times a constant). The fluctuations are confined to the interval $[-\beta, \beta]$ and therefore all iterates of any allowed initial value will lie within one of the intervals which constitute the support of the invariant measure. We have chosen $a = 3.498 \cong a_4$ and $10^{-4} \leqslant \beta \leqslant 0.03$.

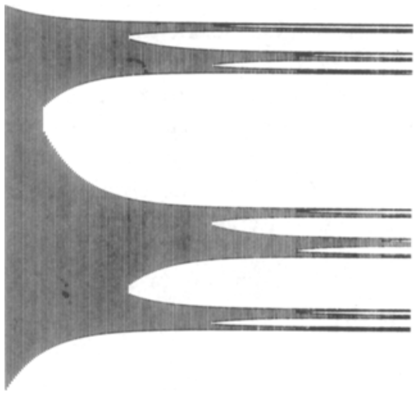

Fig. 2. Same as Fig. 1, but we have chosen $a = 3.569945 \cong a_c$ and $10^{-6} \leqslant \beta \leqslant 0.03$. If the fluctuations go to zero the support tends to the orbit of "period 2^{∞}" which appears to be a "cantoresque" fractal as can be easily seen from this construction.

which the first two columns of Table I belong) Crutchfield and Huberman observed a "bifurcation gap"(24) which is just the result of these noise induced mergence transitions.

It seems that the ratio of successive points β_{i+1}/β_i where mergence transitions take place, tends to a limit which is decreasing for growing parameter values a. (It is approximately 8.0 for 2^{∞} and 6.0 for $3 \cdot 2^{\infty}$; cf. Table I.) This behavior seems to confirm the observation(34) that for the deterministic system (1) at the critical parameter $a = a_c$, the spectral amplitudes of different frequencies corresponding to periods $p = 2^n$ decrease with a factor of 6.6.

Table I. Mergence Transition Points

For the parameters: $a_1 = 3.5697$ (period 64); $a_2 = 3.569945 \cong a_c$ ("period 2^{∞}"); $a_3 = 3.8493 \cong a_{c'}$ ("period $3 \cdot 2^{\infty}$")

a	a_1	a_2	a_3
β_1	1.106×10^{-2}	1.104×10^{-2}	5.945×10^{-4}
β_2	1.267×10^{-3}	1.255×10^{-3}	8.580×10^{-5}
β_3	1.548×10^{-4}	1.484×10^{-4}	1.208×10^{-5}
β_4	2.094×10^{-5}	1.749×10^{-5}	2.176×10^{-6}
β_5	3.691×10^{-6}	2.056×10^{-6}	
β_6		7.533×10^{-7}	
β_7		2.980×10^{-8}	

For the calculations above we evaluated the support of the invariant measure P^* which is at the same time the set R_n of all possible values of $\tilde{x}_n$ obtained from (3) for $n \to \infty$ starting at x_c and allowing all possible sequences ξ_n of fluctuations of a given maximal size β. When we want to determine all possible orbits through x_c we find a sequence of intervals R_n which can be constructed as follows: It is clear that $\tilde{x}_1 = f(x_c) + \xi_1 \epsilon [f(x_c) - \beta, f(x_c) + \beta] := R_1$. From the continuity of the function f we get a recursion formula for the intervals $R_n := [r_n, s_n]$ which is given by

$$r_{n+1} = \min\{f(r_n), f(s_n)\} - \beta$$
$$s_{n+1} = \begin{cases} \max\{f(r_n), f(s_n)\} + \beta & \text{if} \quad x_c \notin [r_n, s_n] \\ f(x_c) + \beta & \text{if} \quad x_c \in [r_n, s_n] \end{cases} \tag{13}$$

The result can be seen in Fig. 1 where we have plotted the support of P^* versus the logarithm of fluctuation size β for the parameter $a_4 = 3.498$ where period 4 is maximally stable for system (1). For $0 < \beta \leqslant \beta_2 \cong 1.6 \times 10^{-3}$ we have semiperiod 4 and at $\beta_1 \cong 1.72 \times 10^{-2}$ there is a mergence transition from semiperiod 2 to semiperiod 1, i.e., to purely aperiodic behavior. In Fig. 2 we have the same plot for $a \cong a_c$ where from an orbit of period 2^∞ for $\sigma = 0$ (deterministic) an infinite sequence of mergence transitions appears if the fluctuations are switched on. Figure 2 also shows that a fractal is constructed if the fluctuations go to zero and the orbit of "period 2^∞" is reached.[31] Namely, the middle part of the initial interval which constitutes the support of the invariant measure for maximal fluctuations is removed at the largest point of mergence transition. Then the middle parts of the remaining two intervals are removed and so on. This is the same

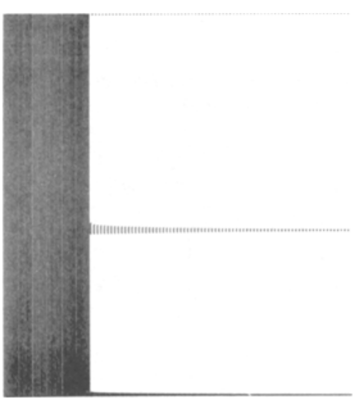

Fig. 3. Same as in Fig. 1, but we have chosen $a = 3.83187 \cong a_3$ and $10^{-4} \leqslant \beta \leqslant 10^{-3}$. For the perturbed period-3 orbit the change of the support is more drastic and occurs at much lower fluctuations than in the previous cases.

procedure as at the construction of a Cantor set but the removed fraction is not constant here. (Note however, that the resulting fractal does not correspond to a strange attractor.)

In Fig. 3 the case $a = a_3 = 3.8318 \cdots$ is quite different. Here we have a transition from (prime) semiperiod 3 to semiperiod 1 at a fluctuation of $\beta \cong 5.6 \times 10^{-4}$.

In our calculations we evaluated 10^7 iterates of the critical point x_c, and at the same time it was recorded how many times the system has been in each one of the subintervals $I_1, \ldots, I_{1000}$ of the unit interval. The choice of the initial point is motivated by the Singer theorem, which, for our case, implies that the system possesses at most one stable periodic orbit which then would be approached by the orbit through x_c.[32] It appears that the histogram is independent of the starting point $\tilde{x}_0$ as long as it is contained in the interval $I_B(\beta)$ of the proposition.

We examined the influence of stochastic fluctuations on the invariant measure of the logistic system mainly for three characteristic parameter values: At $a_4 = 3.498$ we have a (super) stable period 4 for the unperturbed system. In Fig. 4 we added equally distributed fluctuations of standard

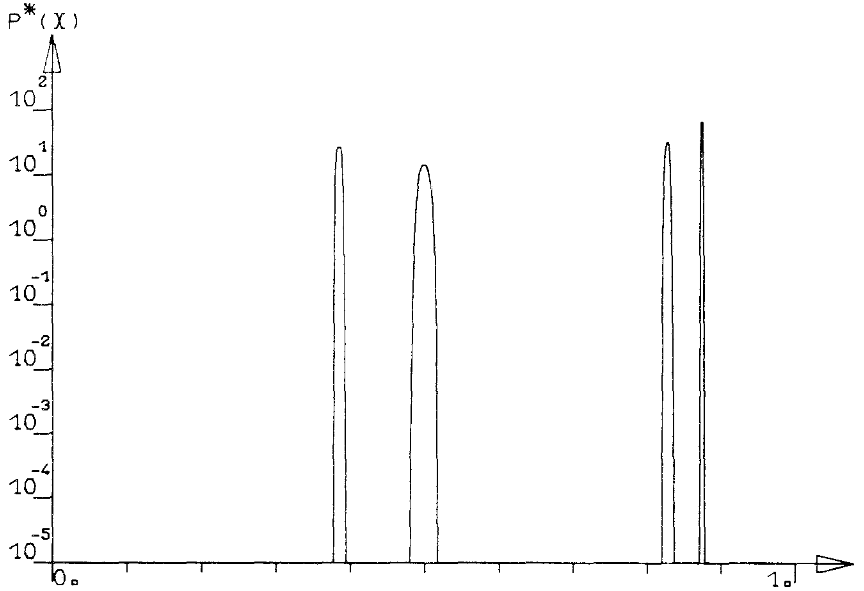

Fig. 4. The invariant measure is approximated by a histogram obtained from 10^7 iterates of the critical point $x_c = 0.5$ and a equipartition of the unit interval into 1000 subintervals. The scale of the vertical axis is logarithmic. [Note that $P^*(x) = 10^{-4}$ means that the system has visited the corresponding subinterval once in 10^7 iterations.] Histogram for $a = 3.498 \cong a_4$ and $\sigma = 10^{-3}$.

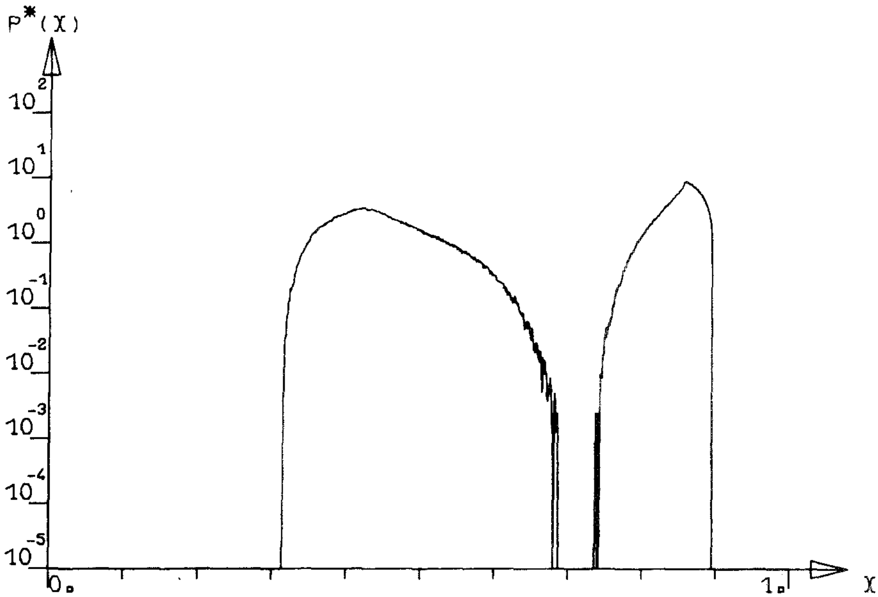

Fig. 5. Same as in Fig. 4 but $\sigma = 0.011$.

deviation $\sigma = 10^{-3}$. Here the histogram is clearly no more "δ-distribution-like" but still concentrated around the periodic points, i.e., we obviously have a semiperiodic orbit of period 4. As we increase σ, two of the peaks overlap such that now the semiperiodic orbit has period 2 (Fig. 5). At still larger values of σ the remaining two broad peaks also overlap until the histogram becomes more or less flat (Figs. 6 and 7). At $a_3 = 3.831\ldots$ the unperturbed system has a (super) stable period-3 attractor. There is also a large neighborhood of a_3 in which period 3 is stable (in fact it is the largest "parameter window" of a stable orbit of period $\neq 2^n$, i.e., for which $a > a_c$). In contrast to the case of a_4 there are now also *aperiodic* orbits present (according to the theorem of Li and Yorke) which are however *unstable*. Here the situation is again (as we have already mentioned for the support of the measure) qualitatively different to the above case: For small fluctuations with $\sigma < 5 \times 10^{-4}$ we again have very narrow peaks (Fig. 8), but at $\sigma = 10^{-3}$ the situation is completely different from the case of a_4. The three peaks are still very narrow but the histogram is now already positive everywhere in the invariant interval $I_a = [f(f(x_c) + \beta) - \beta, f(x_c) + \beta]$.

So it seems that the system escapes from the periodic islands now and then to stay some time on the aperiodic invariant set until it is caught again

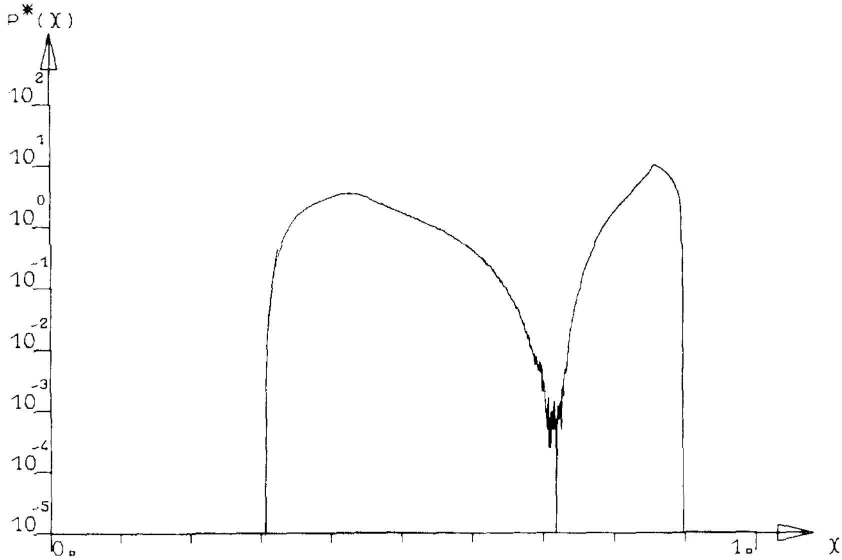

Fig. 6. Same as in Fig. 4 but $\sigma = 0.012$.

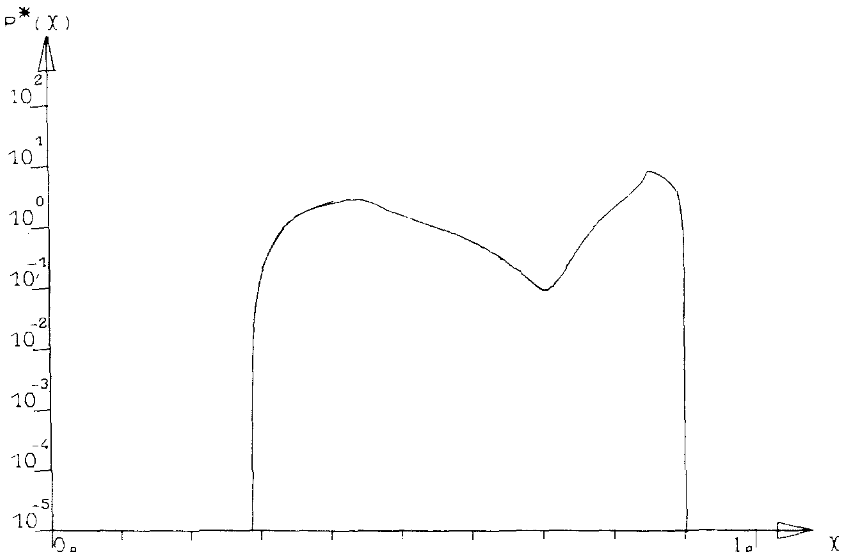

Fig. 7. Same as in Fig. 4 but $\sigma = 0.015$.

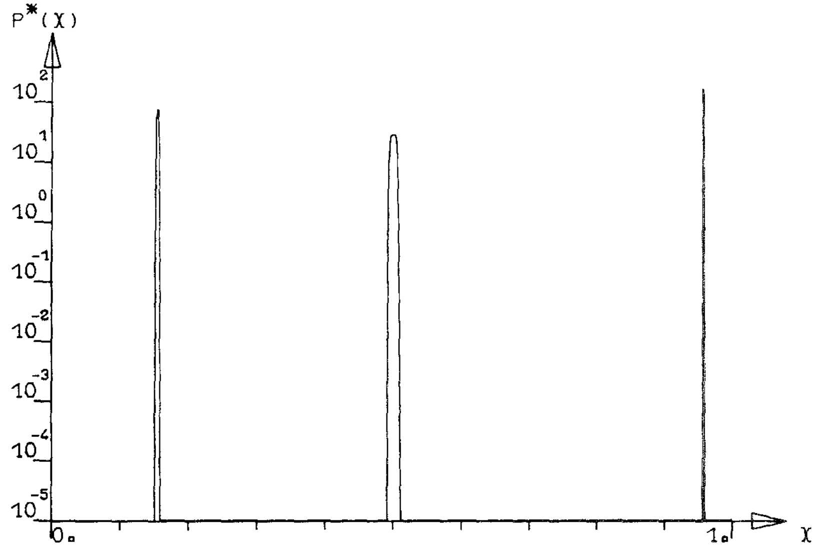

Fig. 8. Same as in Fig. 4 but $a = 3.8318\ldots \cong a_3$ and $\sigma = 3.5 \times 10^{-4}$.

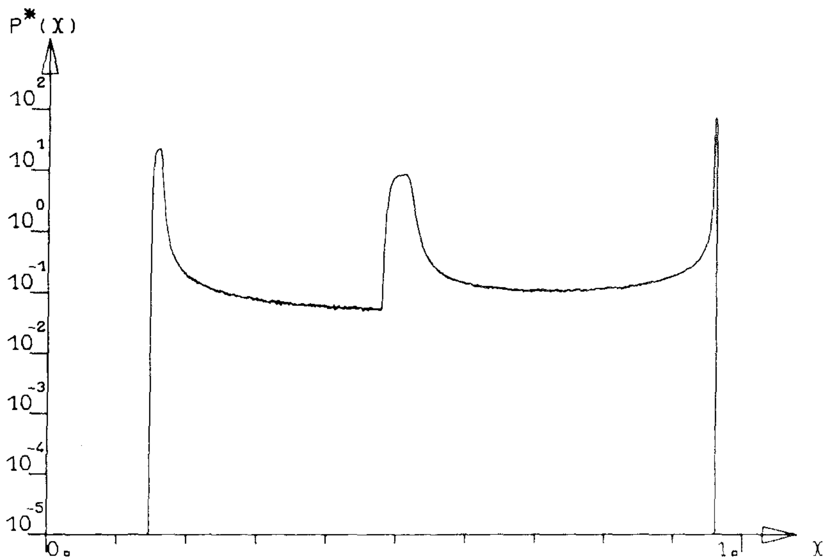

Fig. 9. Same as in Fig. 8 but $\sigma = 10^{-3}$.

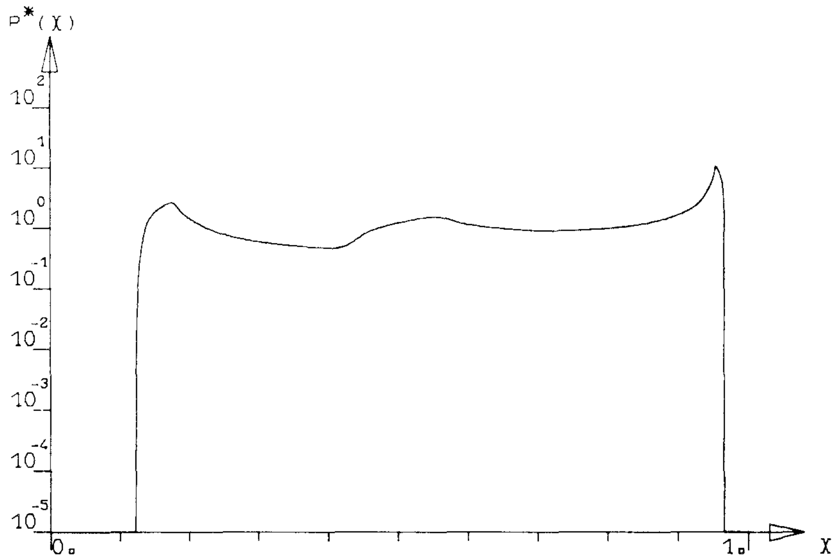

Fig. 10. Same as in Fig. 8 but $\sigma = 4 \times 10^{-3}$.

by the periodic islands (Fig. 9). If we now increase σ to its maximal value, the only change is a broadening of the peaks until the histogram is flat as in Fig. 10 where $\sigma = .004$. For a typical parameter value at which the unperturbed system behaves really chaotic we have chosen $a_u = 3.9$. Here all periodic orbits—as far as numerical experiments can tell—are unstable and so the histogram is already positive in the entire invariant interval for $\sigma = 0$. However, it shows a very large number of narrow peaks which correspond to the orbit through the critical point x_c. For small fluctuations with $\sigma < 10^{-6}$ we could distinguish on the computer the first 25 iterates of x_c. For larger values of σ the peaks which correspond to the highest iterates of x_c become broad and vanish while the other peaks are still very narrow. This is also what one would expect from the increase of the standard deviation of the sum of the fluctuations (Figs. 11 and 12).

6. THE LYAPUNOV CHARACTERISTIC EXPONENTS

A histogram which is everywhere positive in the invariant interval is not a sufficient indicator for a chaotic behavior of the system, so, e.g., the attractor could consist of a quasiperiodic orbit which fills the unit interval in quite a regular manner. In that case nearby initial points would stay

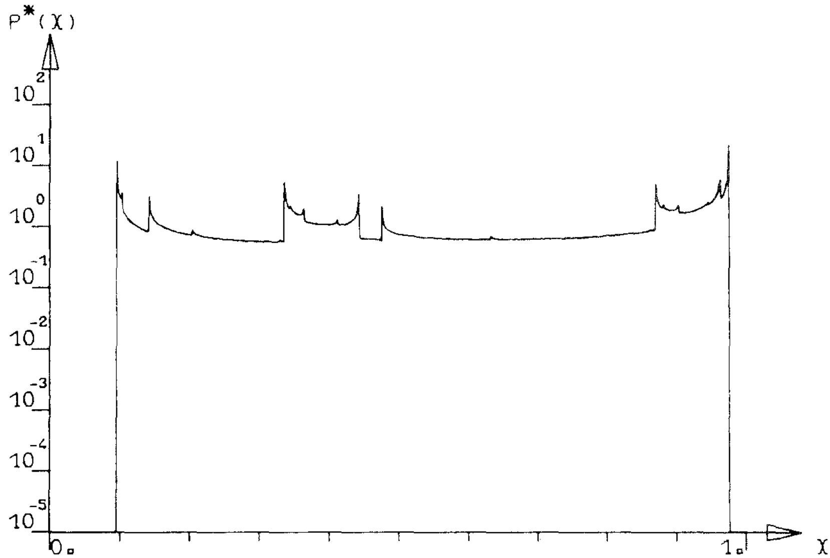

Fig. 11. Same as in Fig. 4 but $a = 3.9$ and $\sigma = 0.0$.

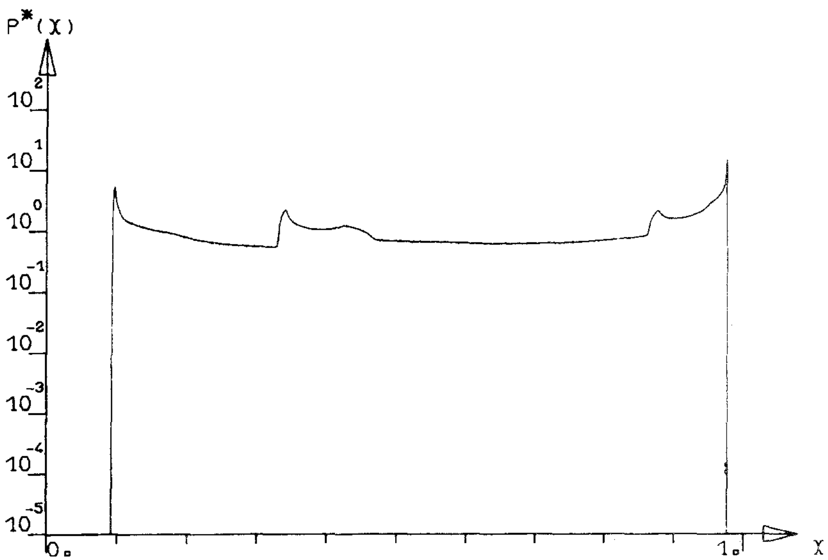

Fig. 12. Same as in Fig. 11 but $\sigma = 10^{-3}$.

together for a very long time. A quantity which measures the average divergence of nearby paths is the Lyapunov characteristic number $\lambda(x_0)$. It is defined in our case [one-dimensional discrete systems (2)] by

$$\lambda(x_o) := \lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} \ln\left| \frac{df}{dx}(x_k)\right| \tag{14}$$

It is also possible to interpret λ as the (information theoretic) entropy of the dynamical system or as its production rate of information (for a more detailed discussion see Ref. 16). For the perturbed system (3) the Lyapunov exponent λ can be defined in close analogy to the formula (14) if we only replace the orbit points x_k by the perturbed sequence $\tilde{x}_k$ of (3). Thus the effect of fluctuations on the Lyapunov exponents would just reflect the different invariant measures P^* from Eq. (8). (In our case the stochastic perturbations ξ_k are independent of x.)

Thus we can replace the time average from Eq. (14) by the ensemble average induced by P^*:

$$\lambda = \int_{-\infty}^{\infty} P^*(x)\ln\left| \frac{df}{dx}(x)\right| dx \tag{15}$$

In Fig. 13 we have a plot of the Lyapunov characteristic exponents versus the relevant parameter in the interesting region $a \in [3.4, 4]$ for the unperturbed system (1).[16]

For many parameters $a > a_c$ the systems which had periodic attractors for $\sigma = 0$ became already chaotic, i.e., with $\lambda > 0$ for $\sigma = 1.5 \times 10^{-4}$ (cf. Fig. 14). However, the Lyapunov exponents of many systems with $a < a_c$ are also positive although we know from the support of the invariant measure that for such a small fluctuation the systems are still semiperiodic of period $n \geqslant 4$ (cf. Table I). This also confirms the observation that the Lyapunov exponents are not sensitive to phase coherence (power spectrum).[28] For $\sigma = 10^{-3}$ all the λ are positive except the ones which correspond to period 3 or to a period 2^n for $n < 3$ (Fig. 15). For a slightly larger value of σ (at about $\sigma \cong 1.2 \times 10^{-3}$ all systems for which $a > a_c$ have a positive λ. The Lyapunov exponents of systems of period 2^n grow only very slowly until they also take on small positive values, e.g., for $a_4 = 3.498$ this happens at $\sigma \cong .018$.

In Fig. 16 we see three typical features of the relations between the Lyapunov numbers and the standard deviation σ of the fluctuations:

1. For $a < a_c$ we have slow growing to small positive values of λ.

2. For $a_p > a_c$, which corresponds to a stable periodic orbit of the unperturbed system, we have very fast increase of λ to large positive values.

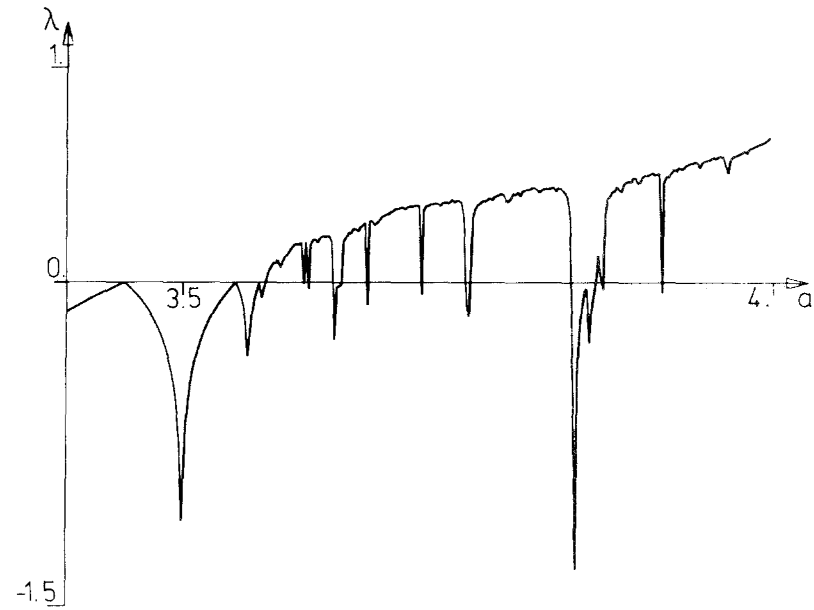

Fig. 13. The Lyapunov exponents were calculated according to formula (14) with the approximation $n > 10^5$ and $x_0 = f(x_c)$. Lyapunov exponents λ versus the parameter $a \in [3.4, 4]$ for $\sigma = 0.0$.

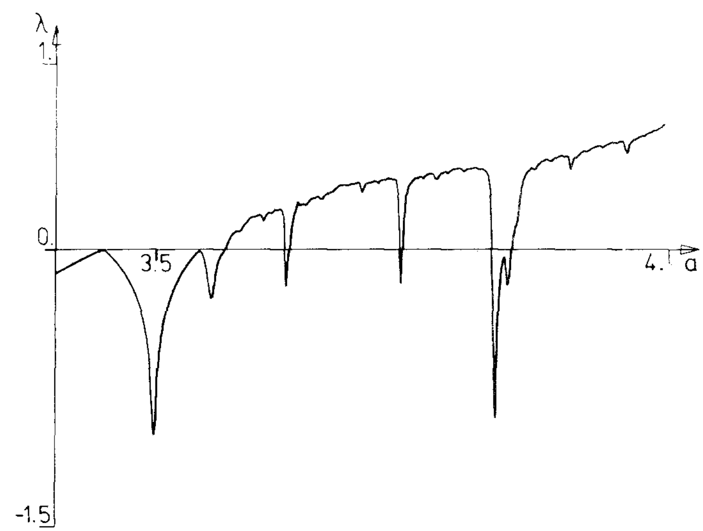

Fig. 14. Same as in Fig. 13 but $\sigma = 1.5 \times 10^{-4}$.

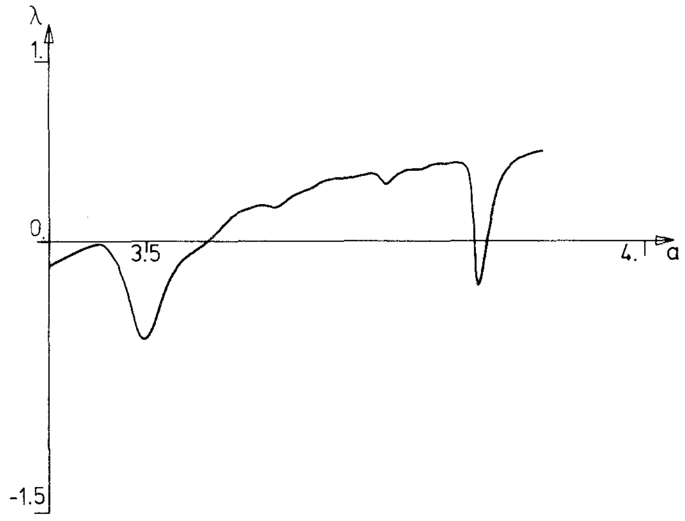

Fig. 15. Same as in in Fig. 13 but $\sigma = 10^{-3}$.

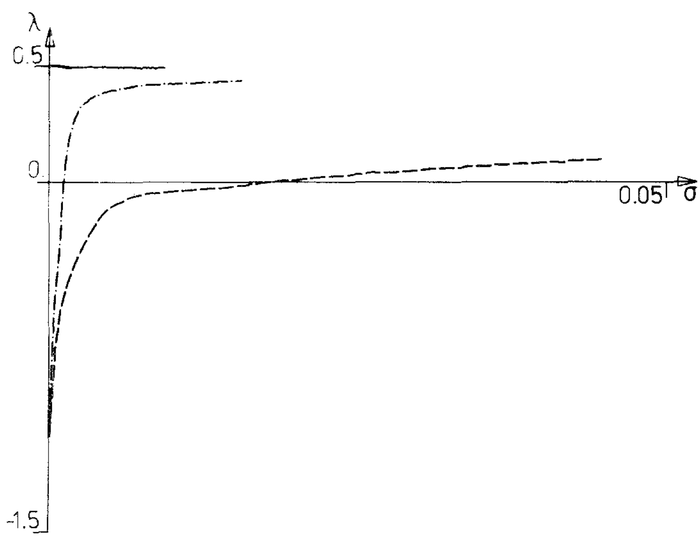

Fig. 16. Lyapunov exponents λ versus fluctuation $\sigma \in [0, 0.05]$ for $a = 3.498 \cong a_4$ (dashed line), $a = 3.8318 \cdots \cong a_3$ (dot-dashed line), $a = 3.9$ a_u (solid line).

3. For $a_u > a_c$, the system has an aperiodic attractor already in the unperturbed case. Here the Lyapunov exponent seems to be independent of σ.

In our calculations we evaluated the λ for $n > 10^5$ iterations and for 100 different values of $\sigma < \sigma_{max}$, where σ_{max} corresponds to β_{max}, which is given by (6) and which can be read off from Fig. 16.

7. AUTOCORRELATION FUNCTION AND POWER SPECTRUM

For periodic orbits the autocorrelation function is also periodic, i.e., with nondecreasing amplitude. The corresponding power spectrum therefore consists only of sharp frequency lines.[33]

If fluctuations are added, we observe a very slow decrease of the correlation function which produces small contributions of broadband noise to the power spectrum.

In Figs. 17, 18, and 19 we have evaluated the power spectrum for the perturbed period-3 orbit for three different values σ. The observed broadening of the spectrum is in full agreement with the above results obtained from the histogram and the Lyapunov exponents.

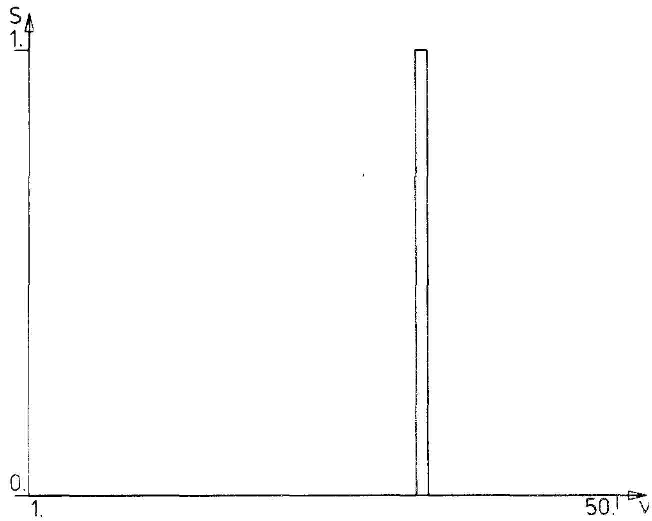

Fig. 17. Power spectrum S versus frequencies $\nu \in \{1, \ldots, 50\}$ for $a \cong a_3$ and $\sigma = 0$. The autocorrelation function was evaluated for time lags up to 101 and averaged over 2000 iterations. Using the fast Fourier transform the power spectrum was then calculated for 50 frequencies. In both cases we used the library functions of digital. All the computations were also performed on a PDP11-machine of digital.

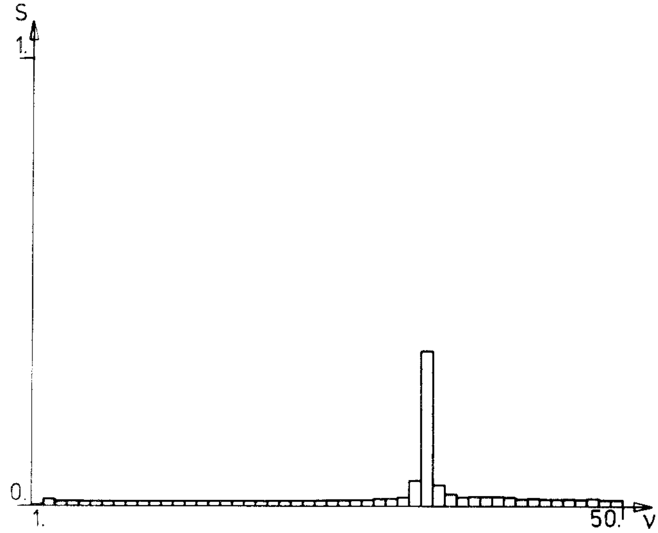

Fig. 18. Same as in Fig. 17, but we have chosen $\sigma = 1.5 \times 10^{-4}$.

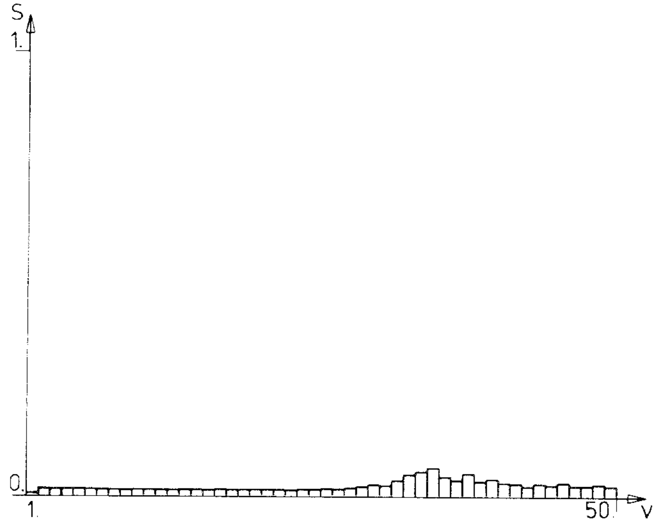

Fig. 19. Same as in Fig. 17, but we have chosen $\sigma = 10^{-3}$.

8. PERIODIC PERTURBATIONS

We also made some calculations where the perturbations were not random but periodic, i.e., we simulated the interaction between the logistic system and a periodic system. The preliminary results show that almost everything can happen: If we perturb period 3 with a system of period 2 we also get a sequence of (subharmonic) bifurcations until the system becomes chaotic. On the other hand we also get chaos if we perturb period 3 by period 3 while we get (semi) periodic behavior if we perturb the logistic system at a parameter value for which it is itself chaotic. So for instance, at $a = 3.9$ a periodic perturbation of period 3 with standard deviation $\sigma = 10^{-2}$ leads to a histogram which consists only of several narrow peaks indicating a very regular behavior. Therefore we conjecture that the qualitative behavior of a chaotic system is stable against fluctuations but not against periodic perturbations. On the other hand systems with a periodic attractor can become chaotic under both kinds of perturbations.

ACKNOWLEDGMENTS

We gratefully acknowledge the valuable discussions we had with C. Gardiner, J. P. Eckmann, and R. H. G. Hellemann. During our work at the computer we got helpful advice from H. Ohno, Th. Harbich, and A. Kuchelmeister. We would also like to thank our colleagues, especially M. Renardy, G. Hentschel, S.Chaturvedi, and K. Zeile, for numerous discussions and their critical interest. We wish to thank the Volkswagen foundation for financial support.

APPENDIX A: PROOF OF THE PROPOSITION

The system (3) possesses the critical point $x_c = 0.5$. From the continuity of f_a it follows that for all $\xi_n \in I_\beta$ the interval

$$I_a := \left[f_a f_a(x_c) + \beta\,) - \beta, f_a(x_c) + \beta \right] \tag{A1}$$

is mapped into itself under (3) if

(i) $$f_a(x_c) + \beta \leqslant 1 \tag{A2}$$

and

(ii) $$f_a(f_a(x_c) + \beta\,) - \beta \geqslant \beta$$

It is easily checked that (ii) implies (i) for $a \in [3,4]$, while (ii) is fulfilled for

$$\beta \leqslant \frac{1}{4a}\left[2a - a^2 - 4 + 2(3a^2 - 4a + 4)^{1/2} \right] \tag{A3}$$

as a straightforward calculation shows. Furthermore there is an interval $I_B(\beta) := [u(\beta), v(\beta)]$ such that for all $\tilde{x}_0 \in I_B(\beta)$ there is an $n_0 \in \mathbb{N}$ such that for all $n > n_0$ the members $\tilde{x}_n$ of the orbit through $\tilde{x}_0$ are contained in I_a. This basin is determined by the condition

$$\begin{aligned} f_a(u(\beta)) - \beta &= u(\beta) \\ f_a(v(\beta)) - \beta &= u(\beta) \end{aligned} \tag{A4}$$

From (A4) we get

$$\begin{aligned} u(\beta) &= \frac{a-1}{2a}\left[1 - \left(1 - \frac{2a\beta}{(a-1)^2}\right)^{1/2}\right] \\ v(\beta) &= \frac{1}{2} + \left(\frac{1}{4} - \frac{u+\beta}{a}\right)^{1/2} \end{aligned} \tag{A5}$$

Since $f(\beta) - \beta \geqslant \beta$ for $\beta \in [0, (a-2)/a]$ we have $u(\beta) \leqslant \beta$ for $a \in [2, 4]$. Thus $\beta_{\max}$ is determined as the largest root of the function $h(\beta) := f_a(f_a(x_c) + \beta) - \beta - u(\beta)$. ■

APPENDIX B: DERIVATION OF THE RECURSION FORMULA (8)

For a given random sequence of perturbations $(\xi_1, \ldots, \xi_n, \ldots)$ where $\xi_i \in \mathbb{R}$ the system (3) will be found after n steps at a certain point $x \in [0, 1]$ with a probability given by

$$p(x, n) = \delta(x - \tilde{x}_n) \tag{B1}$$

where $\tilde{x}_n$ is defined by (3) and the initial distribution was given by

$$p(x, 0) = \delta(x - x_0) \tag{B2}$$

Since we want to take into account all possible realizations of $\xi_i \in \mathbb{R}$ we have to integrate over them at each intermediate step with a certain weight $R(\xi_i)$. Then we get from (B1) the averaged distribution

$$\hat{P}(x, n) = \int_{\mathbb{R}^n} \prod_{i=1}^{n} R(\xi_i)\, d\xi_i\, \delta(x - \tilde{x}_n) \tag{B3}$$

This is identical with $\tilde{P}(x, n)$ of Eq. (8), as can be shown by induction.

Let $\hat{P}(x, 0) := \delta(x - x_0)$, from (B3) and (3) we know that $\hat{P}(x, 1) = R(f(x_0) - x)$, which coincides with the result obtained by inserting $\tilde{P}(x, 0)$ into (8), i.e., we have $\hat{P}(x, 1) = \tilde{P}(x, 1)$. If now the wanted identity holds after $n - 1$ steps, i.e., $\hat{P}(x, n-1) = \tilde{P}(x, n-1)$, we are allowed to insert $\hat{P}(x, n-1)$ into (8) and get

$$\begin{aligned}\tilde{P}(x,n) &= \int_{\mathbb{R}} R(x - f(y)) \int_{\mathbb{R}^{n-1}} \prod_{i=1}^{n-1} R(\xi_i)\, d\xi_i\, \delta(\tilde{x}_{n-1} - y)\, dy \\ &= \int_{\mathbb{R}^{n-1}} \prod_{i=1}^{n-1} R(\xi_i)\, d\xi_i\, R(x - f(\tilde{x}_{n-1})) \\ &= \hat{P}(x,n)\end{aligned}$$

In the last step we integrated in (B3) over ξ_n and used the fact that $\tilde{x}_n = f(\tilde{x}_{n-1}) + \xi_n$.

NOTE ADDED IN PROOF

After submission of this paper for publication, we received a preprint from J. P. Crutchfield, J. D. Farmer, and B. A. Huberman,[35] where further results on the effect of noise on 1-D maps are obtained. J. P. Crutchfield kindly informed us that Y. Oono and Y. Takahashi[36] used Fredholm theory to treat the stability of chaos against external noise. A similar method was also used by S. -J. Chang and J. Wright[37].

In Ref. 38, we showed that the noise induced transition from period 3 to chaos occurs via intermittency. Later on the general theory has been developed by J. -P. Eckmann, L. Thomas, and P. Wittwer,[39] and by J. E. Hirsch, B. A. Huberman, and D. J. Scalapino.[40] Our idea of determining the support of the invariant measure of system (3) (cf. section 5) can be rigorously formulated by using the notion of (ϵ, δ)-diffusions.[41] In Ref. 42 we generalized eq. (8) for higher dimensional systems and for multiplicative noise.

REFERENCES

1. E. N. Lorenz, *J. Athmos. Sci.* **20**:130 (1963).
2. J. Curry, *Commun. Math. Phys.* **60**:192 (1978).
3. H. Haken, *Phys. Lett.* **53A**:77 (1975).
4. H. Haken and A. Wunderlin, *Phys. Lett.* **62A**:133 (1977).
5. O. E. Rössler, *Z. Naturforsch.* **31a**:259 (1976).
6. O. E. Rössler, *Phys. Lett.* **60A**:392 (1977).
7. R. M. May, *Nature* **261**:459 (1976).
8. S. Smale, *Bull. Am. Math. Soc.* **73**:747 (1967).
9. T. Kai and K. Tomita, *Prog. Theor. Phys.* **61**:54 (1979).
10. R. Bowen, *Commun. Math. Phys.* **69**:1 (1979).
11. M. Misiurewicz, *Absolutely Continuous Measures for Certain Maps of an Interval* in *Publications Mathématiques*, No. 53, Institut des Hautes Etudes Scientifiques (Bures-Sur-Yvette), to appear.
12. J. Guckenheimer, *Commun. Math. Phys.* **70**:133 (1979).
13. D. Ruelle, *Commun. Math. Phys.* **55**:47 (1977).

14. T.-Y. Li and J. A. Yorke, *Am. Math. Monthly* **82**:985 (1975).
15. Y. Oono, *Prog. Theor. Phys.* **59**:1028 (1978).
16. R. Shaw, *Naturforsch.* **36A**:80 (1981).
17. S. D. Feit, *Commun. Math. Phys.* **61**:249 (1978).
18. I. Shimada and T. Nagashima, *Prog. Theor. Phys.* **59**:1033 (1978).
19. T. Shimizu and N. Morioka, *Phys. Lett.* **66A**:182 (1978).
20. K. A. Robbins, *SIAM J. Appl. Math.* **36**:457 (1979).
21. P. Collet and J.-P. Eckmann, *Commun. Math. Phys.* **73**:115 (1980).
22. P. Collet, J.-P. Eckmann and O. E. Lanford, *Commun. Math. Phys.*, **76**:211 (1980).
23. G. Mayer-Kress and H. Haken, *Verhandl. DPG(vi)* **15**:360 (1980).
24. J. P. Crutchfield and B. A. Huberman, *Phys. Lett.* **77A**:407 (1980).
25. A. Zippelius and M. Lücke, *J. Stat. Phys.*, **24**:345 (1981).
26. B. Bunow and G. H. Weiss, *Math. Biosci.* **47**:221 (1979).
27. E. N. Lorenz, Noisy periodicity and reverse bifurcation, in *Proceedings of the 1979 Conference on Nonlinear Dynamics*, R. G. H. Hellemann, ed. Annals of the New York Academy of Science.
28. D. Farmer, J. Crutchfield, H. Froehling, N. Packard, and R. Shaw, Power spectra and mixing properties of strange attractors, in *Proceedings of the 1979 Conference on Nonlinear Dynamics*, R. G. H. Hellemann, ed. Annals of the New York Academy of Science.
29. M. Feigenbaum, *J. Stat. Phys.* **19**:25 (1978).
30. S. Grossmann and S. Thomae, *Z. Naturforsch* **32a**:1353 (1977).
31. B. B. Mandelbrot, *Fractals, Form, Chance, and Dimension* (Freeman, San Francisco, 1977).
32. D. Singer, *SIAM J. Appl. Math.* **35**:260 (1978).
33. R. L. Stratonovich, *Topics in the Theory of Random Noise*, Vol. 1, (Gordon & Breach, New York, London, 1963).
34. M. Feigenbaum, *Commun. Math. Phys.* **77**:65 (1980).
35. J. P. Crutchfield, J. D. Farmer and B. A. Huberman, *Fluctuations and simple chaotic dynamics* (submitted to *Physica D*, 1981).
36. Y. Oono and Y. Takahashi, *Prog. Theor. Phys.*, **63**:1804 (1980).
37. S. -J. Chang and J. Wright, *Phys. Rev.*, **A23**:1419 (1981).
38. G. Mayer-Kress and H. Haken, *Phys. Lett.*, **82A**:151 (1981).
39. J. -P. Eckmann, L. Thomas and P. Wittwer, *Intermittency in the presence of noise*, preprint (Geneva 1981).
40. J. E. Hirsch, B. A. Huberman and D. J. Scalapino, *A theory of intermittency*, preprint (1981).
41. D. Ruelle, *Small random perturbations of dynamical systems and the definition of attractors*, preprint (I.H.E.S. 1981).
42. H. Haken and G. Mayer-Kress, *Phys. Lett.*, **84A**:159 (1981).